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Mathematical competencies revisited

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Abstract
This article deals with what it means to possess competence in mathematics. It takes its point of departure in the fact that the notions of mathematical competence and mathematical competencies have gained a foothold as well as momentum in mathematics education research, development and practice throughout the last two decades. The Danish so-called KOM Project (KOM: Competencies and the Learning of Mathematics), the report from which was published in 2002, has played an instrumental role in that development. Since then, a host of new developments has taken place, and we—as the authors of the original report—have felt the need to take stock of this development and revisit the conceptualisation of the basic notions in order to provide an updated version of the original conceptual framework and terminology. Whilst the fundamentals of this framework have been preserved in this article, the version presented here in addition to an up-to-date terminology offers greater clarity and sharpness and richer explanations than found in the original.

Keywords Mathematical competence · Mathematical competencies

1 Introduction

"We should rather enquire who is better wise than who is more wise", says Montaigne (1958, *Of pedantism*, Essays, 1st book, chapter 24).

For almost two decades, and especially since the publication of the so-called KOM report (KOM is an acronym for "Competencies and the Learning of Mathematics") in Denmark in 2002 (Niss & Jensen, 2002), the notions of mathematical competence and mathematical competency/ies have witnessed considerable interest in mathematics education research,

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development and practice in several parts of the world, for a variety of different purposes and in a variety of different contexts. The Project was carried out by a committee appointed in 2000 jointly by the Ministry of Education and the then existing Council for Science Education in Denmark. One of us, Mogens Niss, was asked to chair the committee, whilst Tomas Højgaard Jensen (today Tomas Højgaard, without Jensen), who was a PhD student of Mogens Niss at the time, became the scientific secretary of the committee.

The essential contribution of the KOM report was the introduction and exposition of the concepts of mathematical competence and mathematical competencies with particular regard to their possible roles in the teaching and learning of mathematics. This has generated extensive discussion and a multitude of additional conceptual developments, oftentimes in connection with different theoretical, empirical and practical uses of the notions, as reflected in many publications (e.g., Abrantes, 2001; Alpers et al., 2013; Blomhøj & Jensen, 2003; Blomhøj & Jensen, 2007; Boesen et al., 2014; Catalonian Department of Education, 2013; D'Amore, Godino, & Fandińo, 2008; García et al., 2013; Højgaard, 2009; Højgaard, 2010; Jaworski, 2012; Jensen, 2007; Köller & Reiss, 2013; Leuders, 2014; Lithner, Bergqvist, Bergqvist, & Boesen, 2010; Niss, 1999, 2003a, 2015; Niss & Højgaard, 2011; Romero & Lupiáńez, 2014; Tobón, Prieto, & Fraile, 2010; Turner, Dossey, Blum, & Niss, 2013; Turner, Blum, & Niss, 2015; Undervisningsministeriet, 2014).

In 2016, mathematical competencies and their relatives were the subject of a so-called Survey Team at the Thirteenth International Congress on Mathematical Education (ICME-13), held in Hamburg, Germany. The report of the Survey Team was published in 2016 (Niss, Bruder, Planas, Turner, & Villa-Ochoa, 2016), whilst a short version appeared in the proceedings of ICME-13 (Niss, Bruder, Planas, & Villa-Ochoa, 2017). It is beyond the scope of this article to give an account of the similarities and differences between, on the one hand, the notions of mathematical competence and competencies and, on the other hand, related but not identical notions such as mathematical proficiency, mathematical practices, fundamental mathematical capabilities, mathematical literacy, quantitative literacy and numeracy. Readers are referred to Niss (2015) and Niss et al. (2016) for such accounts.

Against this background, there is good reason to take stock of the development mentioned and to revisit the original conceptualisation of mathematical competence and mathematical competencies. Moreover, accounts of the KOM Project have never before been published in mathematics education research journals, but primarily in various conference proceedings and book chapters (e.g., Blomhøj & Jensen, 2007; Niss, 2003b) in which the primary focus was often not mathematical competencies in and of themselves, but their place and role in some broader context (e.g., Jensen, 2007; Niss, 2015; Niss et al., 2016). So, the purpose of this article is to provide the first systematic journal account of the project, and in that context also to provide an updated conceptual framework of the notions of mathematical competence and competencies, inspired by the developments that have taken place since the early publications in the area. Our research question for the present article is, then, a meta-question:

What would an up-to-date conceptual account of the notions of mathematical competence and of mathematical competencies look like?

Before attempting to answer this question, it is necessary to first clarify the nature of the entire endeavour. As was the case two decades ago, the task we want to undertake today still is to characterise what it means for someone to be mathematically competent. We want to do so in generic terms, i.e., in a manner that is independent of specific mathematical subject matter as well as of specific educational levels. This is a necessary prerequisite for coming to grips with what is *common* to mathematical competence across and beyond subject matter areas and across and beyond education segments, institutions and levels, as well as of fields of mathematical practice, and ultimately for coming to grips with what allows us to use the same term, *mathematics*, for a discipline and a subject which from a purely *subject matter* point of view exists in wildly diverse manifestations with only a very small intersection in terms of content across all levels, ranging from early primary to late tertiary mathematics education.

By focusing on mathematical competence rather than on mathematical subject matter as the integrating factor of mathematics across all its manifestations, we have chosen to focus on the *exercise* of mathematics, i.e., the *enactment* of mathematical activities and processes. Now the obvious question arises of whether it is, at all, possible to identify valid and relevant components of mathematical activity that are common to the enactment and exercise of mathematics wherever these occur. The very existence of this article demonstrates that to us the answer to this question is, of course, “yes”.

Even if this may seem to be a daunting answer, there is inspiration and support to be gained from other fields. Linguistic competence with respect to a given language can be said to consist of four generic components (Gregersen et al., 2003, p. 41): the ability to understand and make sense of other people’s oral speech, the ability to understand and make sense of other people’s written texts, the ability to express oneself orally to make oneself understood and interpreted correctly by listeners, and the ability to express oneself in writing to make oneself understood and interpreted correctly by readers, all of this within different styles, genres and registers. These components of linguistic competence are the same in any language and in all domains across and beyond all education levels, even though pupils in Grade 1 do indeed speak and write about different sorts of things than do PhD students or professors of literature in the language at issue. They also have different vocabularies and different degrees of mastery of orthography and grammar, but the components nevertheless remain the same.

To be sure, by focusing on competence we certainly have no intention to discard the significance and role of mathematical subject matter, including facts, results and methods, in the development and possession of mathematical competence and insight. That would be absurd, just as it would be absurd to discard vocabulary, orthography, grammar and syntax as significant elements in linguistic competence. However, in the same way as it would be absurd to reduce linguistic competence to the mere knowledge of vocabulary, orthography, grammar and syntax, it would be absurd to reduce the ability to exercise and enact mathematics to listing the mathematical concepts, terms, theorems, rules and procedures that people should know. The important thing is not just what you know but how you know it, and what you can do with what you know (cf. Montaigne, 1958, *Of pedantism*, Essays, 1st book, chapter 24).

The fact that mathematical competence can neither exist without mathematical subject matter nor be reduced to a specification of that subject matter suggests that there is an intricate relationship between competence and subject matter that needs to be addressed. We shall do so in Section 6 of this article.

2 Competence and mathematical competence

As an everyday term, competence within some field denotes the ability to master the essential aspects and demands of that field and to act effectively within it on the basis of overview and well-founded judgement.

Competence within a field cannot be realised independently of human beings. Rather, it is a property that people can possess to a larger or smaller extent and put to use in certain sorts of situations and contexts. We therefore suggest the following slightly more focused definition:

Competence is someone's insightful readiness to act appropriately in response to the challenges of given situations.

This definition is an attempt to capture key characteristics of the notion of "competence". In everyday language, "readiness" may be cognitive as well as affective and volitional. However, in line with the approach taken in the KOM Project, here we use this word in a cognitive sense only (see Section 5.1).

The first of these characteristics is that competence is oriented towards action. We use "action" in a broad sense, encompassing physical as well as mental actions, including decision making. Moreover, "readiness to act", in our definition of competence, may also involve a conscious and explicit decision to refrain from undertaking particular actions in a given situation.

Secondly, readiness to act without insight certainly exists amongst human beings but then it is not an instance of competence. On the other hand, competence does not follow alone from being immensely insightful, in case the insights at issue cannot be activated in the broad interpretation of the term "action".

Thirdly, the term "challenge" is a key component in the definition. Depending on the situation and context, challenges can be very diverse in nature, ranging from purely intellectual or scientific, over moral, professional or financial, through to practical challenges. Furthermore, what is a challenge to some people may not be so to others. The same is true of what "meeting the challenge" means and takes. So, challenges—and hence competence—display an inherent duality between subjective and socio-cultural aspects. The degree to which certain actions "meet the challenges" is always a question of who "the judges" are that bring meaning and legitimacy to the actions.

Against this background we are now in a position to define what we mean by mathematical competence, pertaining only, of course, to situations in which (at least some of) the challenges are of a mathematical nature. This gives rise to the following updated[1] definition:

Mathematical competence is someone's insightful readiness to act appropriately in response to all kinds of *mathematical* challenges pertaining to given situations.

It is essential to stress that the "situations" referred to in this definition need not be mathematical in and of themselves, as long as they (may) generate mathematical challenges. As spelled out in the KOM report such situations include a wide variety of intra- or extra-mathematical contexts and situations that actually or potentially call for the activation of mathematics in order to answer questions, solve problems, understand phenomena, relationships or mechanisms, or to take a stance or make a decision, endeavours that give rise to the "challenges" we have in mind in the definition.

3 The approach taken

We were not the first to consider the *enactment* of mathematics—rather than content knowledge and related procedural skills—as the essential constituent in the mastery of

[1]The original definition listed a number of ingredients in the notion of mathematical competence and read (Niss & Højgaard, 2011, p. 49): "Mathematical competence comprises having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role".

mathematics. Already George Pólya (1957/1945, p. V) addressed this issue when he wrote that

> [. . .] a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

Whilst Pólya pays particular attention to mathematical problem solving, Seymour Papert took a slightly more comprehensive view, when he wrote:

> In becoming a mathematician does one learn something other and more general than the specific content of particular mathematical topics? Is there such a thing as a Mathematical Way of Thinking? Can this be learnt and taught? Once one has acquired it, does it then become quite easy to learn *particular* topics—like the ones that obsess our elitist and practical critics? (Papert, 1972, p. 250, italics in the original)

Almost at the same time as Papert, Freudenthal (1973) published his monumental monograph "Mathematics as an Educational Task", which is one big plea for considering mathematics as an activity rather than a body of knowledge only.

Roughly at the same time as the KOM Project was conducted, similar ideas were developed and expressed by US researchers in "Adding It Up" (National Research Council, 2001) and "Mathematical Proficiency for All Students" (RAND Mathematics Study Panel, 2003). The fact that the projects behind these publications were concurrent with the KOM Project means that the thinking in the KOM Project was not influenced by these projects.

As a matter of fact, the KOM framework had two inter-related sources. The first source was analytic introspection by the first author (MN) into the processes of actually doing and using mathematics, based on several decades of experience as a mathematician and a mathematics educator. The second source was the innovative mathematics and science study programmes established at Roskilde University at and shortly after its foundation in 1972 (Niss, 2001). The first author was a protagonist in designing these programmes, and was responsible for the design and implementation of the mathematics programmes at the university. In other words, the KOM framework was not really informed by other frameworks. As will be elaborated in the next section, the framework took its point of departure in the observation that the enactment of mathematics is to do with posing and answering different sorts of mathematical questions.

4 Mathematical competencies

Whilst the concept of mathematical competence focuses on the enactment of mathematics rather than on mathematical subject matter, not much has been said by merely stating that definition. We have to flesh out the kinds of situations and the associated challenges that mathematical competence should enable one to deal with, and also the nature and characteristics of the actions to be undertaken as well as of the insightful readiness involved in undertaking them. In other words, we have to provide a specification of the main constituents of mathematical competence.

We have chosen to use the term *mathematical competency*, with a "y" at the end, to denote a main constituent of *mathematical competence* with an "e" at the end. More specifically, we define:

A** mathematical competenc**y is someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations.

So, whilst "mathematical competenc*e*" involves the activation of mathematics to deal with all sorts of challenges of a situation or context, "a mathematical competenc*y*" focuses on the activation of mathematics to deal with a specific sort of challenge that actually or potentially calls for "specific kinds of activation" of mathematics in order to answer questions, solve problems, understand phenomena, relationships or mechanisms, or to take a stance or make a decision. In other words, in our conceptualisation, mathematical competence is an edifice constituted by a set of mathematical competencies.

Of course, what is to be considered a "main constituent" of mathematical competence is not self-explanatory, not even well-defined. When looking for the "right kinds" of competencies, there are two ditches that we should avoid falling into. The first is to provide insufficient further substantiation and specification of the notion of mathematical competence, which would amount to simply restating the definition in some form or another. The other—opposite—ditch to be avoided is to provide too much detail in the specification, for example by way of long lists of elements, making it impossible to see the forest for all its trees. Striking a balance that will keep us away from both ditches requires a combination of analytic power and clarity as well as pragmatism. One or two competencies would be too few, 20 too many. We wanted to construct a set of competencies that would cover and span the entirety of mathematical activity. At the same time, we wanted the set to be convincing, clear and easy to grasp without jeopardising the complexity of the endeavour. The competencies arrived at resulted from a theoretically and experientially based analysis of mathematical activity in its entirety.

In order to determine the sorts of challenges and associated ways of activating mathematics that are to be considered main constituents in mathematical competence, we first offer a brief outline of the purposes, the nature and the roles of mathematical activity. As stated above, an overarching purpose of mathematical activity is to "pose and answer questions in or by means of mathematics". The ability to effectively engage in posing and answering such questions has four different components (to be further detailed below): fundamental mathematical thinking; posing and solving mathematical problems; dealing with mathematical models and modelling; undertaking mathematical reasoning. Moreover, the ability to carry out mathematical activity further involves mastering mathematical language, constructs and tools, again having four different components (also to be further detailed below): dealing with mathematical representations; dealing with mathematical symbols and formalism; undertaking mathematical communication; dealing with material mathematical aids and tools.

What has been altered in this article compared to the earlier versions of the competency framework is the exact wording of definitions, the specific division of labour between the different competencies, the more detailed descriptions and explanations of the individual competencies, as well as a number of aspects of terminology. As it would be to go too far in a journal article to spell out the multitude of changes to the original versions of the framework that this exposition represents, we shall restrict ourselves to pointing out the most significant changes in the exposition to follow. However, the "remarks" paragraphs, which further explain and exemplify the characterisation of each competency, have all been re-written in order to capture conceptual developments that have occurred since the first publication of the framework in 2002.

4.1 Posing and answering questions in and by means of mathematics

As mentioned, there are four competencies in the first category.

4.1.1 Mathematical thinking competency—engaging in mathematical inquiry

This competency involves being able to relate to and pose the *kinds* of generic questions that are characteristic of mathematics and relate to the nature of answers that may be expected to such questions. It further involves relating to the varying scope, within different contexts, of a mathematical concept or term, as well as distinguishing between different types and roles of mathematical statements (including definitions, if-then claims, universal claims, existence claims, statements concerning singular cases, and conjectures), and navigating with regard to the role of logical connectives and quantifiers in such statements, be they propositions or predicates. Finally, it involves relating to and proposing "abstractions" of concepts and theories and "generalisation" of claims (including theorems and formulae) as processes in mathematical activity.[2]

Remarks. Generic questions in mathematics include "does there exist ...?", "if so, under what conditions?", and "how many?", "if an object possesses property A will it necessarily possess property B as well?", "is it possible that ...?", "will this conclusion hold under weaker assumptions?", "does the inverse implication hold as well?" and so on and so forth. Oftentimes, the scope of a concept (e.g., "number", "angle" or "function") gets expanded when the domain in which the concept was first introduced is enlarged (e.g., from natural numbers to rational, real or complex numbers, from plane geometry to 3D geometry or vector spaces, from functions defined by explicit algebraic expressions to functions defined in set-theoretic terms).

Finally, it is worth noting that processes of specific mathematical reasoning are not placed in this competency but are placed in a separate competency of their own, see below.

4.1.2 Mathematical problem handling competency—posing and solving mathematical problems

This competency involves being able to pose (i.e., identify, delineate, specify and formulate) and to solve different kinds of mathematical problems within and across a variety of mathematical domains, as well as being able to critically analyse and evaluate one's own and others' attempted problem solutions. A key aspect of this competency is the ability to devise and implement strategies to solve mathematical problems.[3]

Remarks. It is inherent in the notion of "problem" that it requires more than the immediate employment of approaches, methods and procedures that are routine to the problem solver. Since what to one person is a problem may sometimes be a standard task to someone else, the notion of problem is relative to the person attempting to solve it. This competency focuses on mathematical problems, i.e., problems that have been posed within some mathematical universe. It may happen that a problem has arisen from extra-mathematical needs

[2] The explicit emphasis on "engaging in mathematical inquiry" is a change from the original version.

[3] The characterisation of this competency has been changed from the original version, which included both pure and applied mathematical problem handling, thus blurring the delineation between this competency and the modelling competency. In its present version, the problem handling competency deals with intra-mathematical problems only. Another change from the original version is the inclusion of an emphasis on the strategic aspects of problem solving.

or domains, i.e., by way of mathematical modelling. However, it is only problems in their mathematical instantiations that are covered by this competency, whereas the processes of generating mathematical issues from extra-mathematical contexts and situations are placed under the modelling competency right below (cf. Højgaard, 2010). Solving mathematical problems that have arisen in extra-mathematical contexts, but where the actual formulation of the mathematical problems has already been completed, prior to the solving activity proper, is sometimes called "applied mathematical problem solving".

4.1.3 Mathematical modelling competency—analysing and constructing models of extra-mathematical contexts and situations

This competency focuses on mathematical models and modelling, i.e., on mathematics being put to use to deal with extra-mathematical questions, contexts and situations. Being able to construct such mathematical models, as well as to critically analyse and evaluate existing or proposed models, whilst taking purposes, data, facts, features and properties of the extra-mathematical domain being modelled into account, are the core of this competency. It involves relating to and navigating within and across the key processes of the "modelling cycle" in its various manifestations (e.g., Blomhøj & Jensen, 2003; Blum & Leiß, 2007; Niss, 2010).

Remarks. Dealing with mathematics-laden situations that refer to extra-mathematical contexts in a formal manner only, without requiring attention being paid to extra-mathematical features, might be said to involve modelling in a formal sense, by definition. However, since such an undertaking does not implicate the mathematical modelling competency proper, it is not included here but is referred to the previous competency. The modelling competency also involves handling mathematical problems generated by the mathematisation part of the modelling process. Only in such cases is problem handling considered part of the modelling competency.

4.1.4 Mathematical reasoning competency—assessing and producing justification of mathematical claims

The core of the mathematical reasoning competency is to analyse or produce arguments (i.e., chains of statements linked by inferences) put forward in oral or written form to justify mathematical claims. This competency involves both constructively providing justification of mathematical claims and critically analysing and assessing existing or proposed justification attempts. The competency deals with a wide spectrum of forms of justification, ranging from reviewing or providing examples (or counter-examples) over heuristics and local deduction to rigorous proof based on logical deduction from certain axioms.[4]

Remarks. In contradistinction to what is the case with the mathematical thinking competency, the reasoning competency deals with the ability to analyse and carry out specific reasoning meant to provide justification for mathematical claims. Whilst such reasoning does indeed make intensive use of logic it goes far beyond logic by also implicating mathematical substance. It is important to stress that the kinds of claims at issue in this competency are not confined to "theorems" or "formulae" but comprise all sorts of conclusions obtained by mathematical methods and inferences, including solutions to problems.

[4]In comparison with the original characterisation of this competency, which primarily focused on different forms of reasoning, the present one puts an added emphasis on reasoning as justification in the forefront of the description.

4.2 Handling the language, constructs and tools of mathematics

The second category, handling the language, constructs and tools of mathematics, also contains four competencies.

4.2.1 Mathematical representation competency—dealing with different representations of mathematical entities

This competency consists of the ability to interpret as well as translate and move between a wide range of representations (e.g., verbal, material, symbolic, tabular, graphic, diagrammatic or visual) of mathematical objects, phenomena, relationships and processes, as well as of the ability to reflectively choose and make use of one or several such representations in dealing with mathematical situations and tasks. This competency also involves relating to the scopes and limitations—including strengths and weaknesses—of the representations involved in given settings.

Remarks. Different representations of the same mathematical entity are not necessarily equivalent in the sense that they contain exactly the same information about the entity. Therefore, changing between different representations is seldom a “bijective” translation but typically gives rise to information gain or information loss. Paying attention to such gains and losses is a significant element in this competency. One might further say that the three competencies presented below deal with particular kinds of representations, symbolic, verbal or material. However their main emphasis is on dealing with one category of representation at the time, whereas the present competency focuses on the whole spectrum of representations and translations between them.

4.2.2 Mathematical symbols and formalism competency—handling mathematical symbols and formalisms

The ability to relate to and deal with mathematical symbols, symbolic expressions and transformations, as well as with the rules and theoretical frameworks (formalisms) that govern them, constitutes the key component of this competency. On the receptive side, this competency is to do with decoding and interpreting instances of symbolic expressions and transformations, as well as formalisms, already present, whereas the constructive side focuses on introducing and employing symbols and formalism in dealing with mathematical contexts and situations.

Remarks. Whilst the representation competency addresses the relationships amongst and translations between a multitude of representations of mathematical entities, including symbolic representations, the present competency concentrates on the specifics of mathematical symbols and formalism.

4.2.3 Mathematical communication competency—communicating in, with and about mathematics

An individual’s ability to engage in written, oral, visual or gestural mathematical communication, in different genres, styles, and registers, and at different levels of conceptual, theoretical and technical precision, either as an interpreter of others’ communication or as an active, constructive communicator, constitutes the core of this competency.

Remarks. Any communication in, with and about mathematics takes place within frameworks of communication at large but involves significant elements of a specifically mathematical nature that go beyond general communication. Thus, mathematical communication oftentimes invokes mathematical notions and concepts, terms, results and theories, or other features of mathematics as a discipline and a subject, and often involves the use of one or more mathematical representations.

4.2.4 Mathematical aids and tools competency—handling material aids and tools for mathematical activity

This competency focuses on dealing with material aids and tools for mathematical activity, ranging from concrete physical objects and instruments, over specially designed papers and charts, to a wide spectrum of digital technologies designed to represent and facilitate mathematical work. An individual's ability to put such aids and tools to constructive use in mathematical work, as well as to critically relate to one's own and others' use of such aids and tools, constitutes the core of this competency, which also involves paying attention to the affordances and limitations of different mathematical aids and tools and choosing between them on that basis.

Remarks. For millennia mathematics has made use of material aids and tools for its undertakings, from carved bones, counting pebbles (calculi), blocks, cords, rulers, compasses, abaci, slide rulers, mechanical instruments and machines, calculators, computers, tablets, smart phones and so on and so forth. Such aids and tools offer particular kinds of material representations of mathematical objects and processes, which typically require separate and particular theoretical and practical introduction and training before being put to use. This, together with the fact that by being artefacts such aids and tools have many physical properties without any bearing on mathematics, gives rise to particular challenges for being able to deal with them in a thoughtful manner in mathematical contexts and situations.

5 Some comments

5.1 The competencies are of a cognitive nature

Needless to say, individuals' affective, dispositional and volitional traits greatly influence their mathematical learning at large as well as their development and exercise of mathematical competencies. Since a person's affective, dispositional and volitional traits are multivariate functions of multitudes of background variables, life trajectory and experiences produced within and outside schooling and education, these traits are highly individualised. For that reason, and in order to maintain analytical clarity, we have decided to omit affective, dispositional and volitional factors from the notion and definition of mathematical competencies. In other words, in our conceptualisation, competencies are of a purely cognitive nature, as is the "readiness to act" forming part of the generic definition of them. By "readiness", we understand an individual's cognitive prerequisites for engaging in certain activities. This is in contrast to "disposition", which we understand as dealing with affective, attitudinal and volitional traits of the individual with respect to the activities at issue.

It is certainly possible to conceptualise competencies to include or add affective, dispositional and volitional components, as has been done in some places (cf. Niss et al., 2016). However, as this puts great demands on maintaining an analytic distinction between these components and the cognitive ones, we have refrained from including them in this framework.

5.2 The competencies are distinct, but not disjoint

It is important to observe that the competencies are not at all (meant to be) disjoint. On the contrary, each competency overlaps each of the other seven competencies. Yet they are distinct in the sense that each competency has a well-defined identity which singles it out from the other competencies. When one of the competencies is in focus of attention, some or all of the other competencies may enter the stage in auxiliary roles, depending on the situation and context.

As appears from Fig. 1, we have chosen to represent these facts and relationships by way of the so-called *competency flower*, in which each petal is distinct from but overlaps every other petal, even to the extent that there is a non-empty intersection of all the petals in the centre of the flower. The identity of the individual petal is emphasised by the varying intensity of its colour, which is maximum in the core part of the petal whilst fading slightly away towards the boundary and the centre.

5.3 The receptive and constructive facet of each competency

Each competency involves a duality between what we might call its "receptive facet" and its "constructive facet", respectively.

In the receptive facet of a competency, the focus is on the individual's ability to relate to and navigate with respect to considerations and processes which have already been introduced (typically by others) into a given context or situation. For example, one may think of following and assessing an alleged mathematical proof or a proposed solution to a mathematical problem.

In the constructive facet of a competency, the focus is on the individual's ability to independently invoke and activate the competency to put it to use for constructive purposes in given contexts and situations. Examples include devising a mathematical proof of a statement, finding and implementing a way to solve a mathematical problem, reducing a

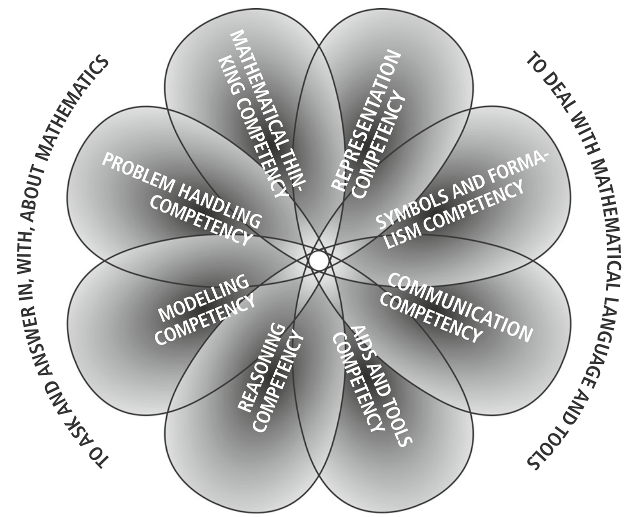

Fig. 1 A visual representation of the eight mathematical competencies (adapted from Niss & Jensen, 2002, p. 45)

complex symbolic expression to an as simple as possible equivalent expression, or actively communicating one's building of a mathematical model.

5.4 Competencies versus procedural skills

If we define a *procedural skill* in mathematics as someone's ability to perform, with accuracy and certainty, a particular, methodologically well-defined—oftentimes algorithmic—goal-oriented type of undertaking, "procedural skill" clearly is a much *less complex concept* than "competency" (Højgaard, 2009). However, exercising a given competency typically requires the activation of a multitude, probably hundreds, of different, very specific procedural skills, each of which draws upon a reservoir of factual knowledge. For example, the symbols and formalism competency involves the procedural skill of performing rule-based transformations of algebraic expressions in different mathematical domains, or determining the derivatives of combinations of standard functions.

Nevertheless, it is important to keep in mind that possessing a given competency involves much more than just the sum of the procedural skills it draws upon. Metaphorically, we might think of a specific procedural skill (as defined here) as an "atom", whereas a competency can be thought of as a huge "molecule" (say a "polymer") composed of a very large number of atoms in different positions. In the same way as the properties of a large molecule are not solely determined by the properties of the individual atoms it contains, a mathematical competency cannot be defined alone in terms of the procedural skills (as defined above) that it entails and draws upon. Moreover, a given set of procedural skills may be pertinent to several different competencies. In summary, procedural skills are necessary but not sufficient for the exercise of a given competency.

5.5 Competencies versus factual knowledge

The core of a mathematical competency is the enactment of mathematics in contexts and situations that present a certain kind of challenge. In other words, a competency focuses on aspects of actively *doing mathematics*. In contradistinction, *knowing mathematics* means to be in possession of factual knowledge about mathematical definitions, concepts, results (propositions and theorems), algorithms, formulae, theories and so on and so forth. Of course, it is impossible to possess a mathematical competency without also possessing at least some factual knowledge. For example, the representation competency requires factual knowledge of the structure and properties of 2D coordinate systems, and of how graphs of, say, linear functions are represented in such a system. Conversely, it is almost impossible to possess factual knowledge without also possessing a minimum of mathematical competence.

Nevertheless, the fact that competencies and factual knowledge are interdependent does certainly not imply that they are equivalent, let alone identical. Their foci are very different, but by no means in conflict.

5.6 Competencies versus understanding

A pertinent and often posed question is: What is the relationship between mathematical competencies and mathematical understanding? A comprehensive answer to this question requires an extensive analysis of the meaning and nature of mathematical understanding going far beyond the scope of this article. So, here, a brief answer has to suffice.

As we see it, it is possible to possess both a deep and a wide understanding of, for example, mathematical concepts, relationships, results, methods, theories and proofs, stored in the mind in a "passive" way, without possessing much mathematical competence, because mathematical competence focuses on the enactment of mathematics. An analogy might illustrate our point: It is possible to possess a wide and deep understanding of chemical elements, substances, phenomena and theories without being able to use chemical equipment and instruments, or being able to carry out and interpret analyses or syntheses, or to design and implement chemical experiments and so on and so forth, which are crucial ingredients in chemical competence.

On the other hand, solid possession of mathematical competence necessarily presupposes a fair amount of mathematical understanding (albeit not necessarily in a very articulate form). In other words, while re-iterating the need for an in-depth analysis, we perceive mathematical understanding as a proper subset of mathematical competence.

5.7 Possessing a competency—three dimensions

As a mathematical competency is manifested and exercised in contexts and situations, there is no such thing as completely and exhaustively possessing it. There is no end to the variety of contexts and situations in which a competency can be activated or to the ways in which it can manifest itself and be set in motion. So, full mastery of a competency is a "point at infinity".

More specifically, we have identified three dimensions in an individual's possession of a competency. The first is the *degree of coverage* of the competency as possessed by the individual. The degree of coverage of a competency is the extent to which all the aspects that define and characterise the competency form part of that individual's possession of the competency. For example, let us consider one of the competencies, the representation competency, say. If an individual's possession of this competency encompasses only the ability to understand and interpret standard representations (e.g., symbolic, graphic or tabular representations of functions) but not the ability to translate and switch between them or the ability to independently introduce and activate them when working mathematically, the degree of coverage of this competency with that individual is smaller than the degree of coverage with someone who can also translate and switch between them.

The second dimension is the *radius of action* of a competency as possessed by the individual. The radius of action represents the range and variety of different contexts and situations in which the individual can successfully activate the competency. Thus, for an individual who can activate the representation competency when dealing with situations that call for, say, the translation between different representations of data about population growth but is unable to deal with contexts and situations that call for similar translations between representations involving national economic or climate data, the radius of action of this competency is smaller with that individual than with someone who can handle data from all kinds of contexts and situations.

The third and last dimension of possession of a competency is the *technical level*, which denotes the level and degree of sophistication of the mathematical concepts, results, theories and methods which the individual can bring to bear when exercising the competency. Thus, an individual who can activate the representation competency only when dealing with, say, familiar standard representations of functions of one variable possesses this competency at a lower technical level than does someone who can also deal with representations of multivariate functions and with representations of statistical data such as histograms, cumulative distribution functions or boxplots.

In case one or more of these dimensions of possession of a competency are missing (take the "value zero") with an individual, that individual does not possess the competency at all. The three dimensions of competency possession can be used to define and characterise progression in an individual's competency possession (an example with respect to the mathematical modelling competency is offered by Jensen, 2007). Such progression simply means that the individual experiences and demonstrates an increase with respect to at least one of these dimensions and no decrease in any of them.

Even though the three dimensions just outlined are formulated by using words with somewhat quantitative connotations ("degree", "radius", "level", "zero"), they are actually meant to be of a qualitative nature, where each of them represents a partial ordering of competency possession: "more (or less) than . . . with respect to . . .".

6 The relationship between competencies and mathematical subject matter areas

By definition and by design, the mathematical competencies go across and beyond specific mathematical subject matter areas, and hence cannot be subsumed under such areas. Does this mean that the competencies have nothing to do with mathematical subject matter? Of course not. Individuals develop mathematical competencies by working mathematically, making use of hosts of mathematical subject matter areas occurring in various manifestations in different contexts, settings and situations. Moreover, an individual's mathematical competence and competencies are activated and exercised in dealing with contexts and situations that actually or potentially contain or involve mathematical subject matter.

Now, if mathematical competencies are not defined in relation to mathematical subject matter areas, whilst, at the same time, they cannot be exercised without involving such areas, what, then, is the nature of the relationship between competencies and subject matter areas? The answer the KOM Project offers to this question is that competencies and subject matter areas constitute two independent but interacting dimensions of mastery of mathematics. In the same way as the competencies cannot be derived from subject matter areas, these cannot be derived from the competencies but have to be determined and selected on other grounds.

We can represent this relationship by a matrix structure. At a given educational level for which $a_1,\ldots, a_n$ have been selected to constitute the subject matter areas considered, the relationship between competencies and subject matter areas can be depicted as in Fig. 2.

Each of the cells in this matrix can be conceptualised in two ways. First, as an answer to the following question: At the educational level under consideration, what is the specific role of competency i $(i = 1, 2,\ldots, 8)$ in dealing with subject matter area j $(j = 1, 2,\ldots, n)$ at this level? Secondly, as an answer to the following question: What is the specific role of subject matter area j $(j = 1, 2,\ldots, n)$ in the activation of competency i $(i = 1, 2,\ldots, 8)$, at the level under consideration?

It is important to stress that this matrix conceptualisation is an "analytic tool" for capturing the principal relationship between mathematical competencies and subject matter areas. It is not meant to be a normative instrument for curriculum design in which the curriculum is being specified by specifying the content of each of the cells (i, j). Nor is it meant to suggest that the relationship between competencies and subject matter areas must be specified by filling in all $8\,n$ cells.

For example, by considering a given competency across the set of subject matter areas, i.e., the row in the matrix corresponding to this competency, it is possible to chart the role of the individual competency across the entire subject matter domain set for the educational

Competency \ **Subj. matter area**	Area 1	Area 2	⋯	Area *n*
Mathematical thinking				
Problem handling				
Modelling				
Reasoning				
Representation				
Symbols and formalism				
Communication				
Aids and tools				

Fig. 2 A matrix structuring of the competencies × subject matter area of mathematics education (adapted from Niss & Jensen, 2002, p. 114)

level at issue. Dually, by considering a given subject matter area across the set of competencies, i.e., the corresponding column of the matrix, it is similarly possible to chart the role of the particular subject matter area in the exertion of the entire set of competencies.

When the competencies are used for educational purposes (see Section 8), it may sometimes be beneficial to use a version of this two-dimensional structure without any cells at all, as in Fig. 3. Longitudinal Danish research and development projects (Højgaard, Bundsgaard, Elmose, & Sølberg, 2010; Ishøj Kommune, 2017; Sølberg, Bundsgaard, & Højgaard, 2015) focusing on this issue have shown that teachers sometimes interpret the matrix structure as a quite stress-generating expectation of the need to enact all $8\,n$ combinations in

Competency \ **Subj. matter area**	Area 1	Area 2	⋯	Area *n*
Mathematical thinking				
Problem handling				
Modelling				
Reasoning				
Representation				
Symbols and formalism				
Communication				
Aids and tools				

Fig. 3 An open two-dimensional structuring of the competencies × subject matter area of mathematics education (adapted from Højgaard, 2012, p. 6416)

their teaching plans, whereas a more open two-dimensional structure can be developed as an invitation for teachers and other educators to introduce their own structure according to the purpose pertaining to the specific circumstances. The open structure can act as a challenging but fruitful framework for determining and representing the content and ambitions of a specific context within mathematics education (Højgaard, 2012).

7 Overview and judgement with regard to mathematics as a discipline

Mathematical competence and the mathematical competencies that span it pertain to an individual's enactment of mathematics in situations that give rise to different sorts of challenges for that individual. However, mathematics is also a scientific discipline which explicitly or implicitly is presented to learners of mathematics in the education system. We know that possessing knowledge and a comprehensive view of mathematics as a discipline does not suffice to generate mathematical competencies in the individual learner and knower of mathematics. Conversely, whilst the enactment of mathematics in a variety of challenging situations, and the exercise of mathematical competencies do of course represent and draw upon crucial aspects of mathematics as a discipline, these aspects do not in and of themselves provide learners and practitioners with a structured and coherent knowledge about and image of mathematics as a discipline.

In the KOM Project, we wanted to include insights into essential features of mathematics as a discipline in our notion of mastery of mathematics. The following three kinds of overview and judgement regarding mathematics as a discipline were identified as crucial to mathematics in its relationships with nature, society and culture. They are described in terms of significant crucial issues.

7.1 The actual application of mathematics within other disciplines and fields of practice

Mathematics is widely used for extra-mathematical purposes in a large variety of everyday, occupational, societal, scholarly and scientific undertakings. This use is brought about by the explicit or implicit construction or utilisation of mathematical models. Exactly which people are in fact using mathematics? When, and in what contexts and situations do they use it and for what purposes? In what ways do they use it, and what are the competencies they possess and activate for so doing?

7.2 The historical development of mathematics, seen from internal as well as from socio-cultural perspectives

Irrespective of which philosophical position one might take on the nature of the relationship between mathematics and reality, it is an undeniable fact that mathematics has developed as a discipline over numerous millennia. It is also a fact that mathematics was sometimes and in some respects developed in close interaction with external needs within other disciplines and fields of practice, and sometimes and in some respects in "splendid isolation" whilst pursuing its own internal goals and serving its own theoretical needs. What are the forces and mechanisms behind the historical development of mathematics in society and culture? In what respects and under what conditions and circumstances is the development of mathematics primarily influenced by internal forces, respectively by external forces?

7.3 The nature of mathematics as a discipline

As a scientific discipline mathematics shares some properties with other disciplines whereas several other significant properties are peculiar and specific to mathematics, in particular the ways in which mathematics obtains and justifies its results. What exactly are the properties that mathematics has in common with other fields and disciplines, what are the properties that are peculiar to mathematics, and what are the causes for the similarities as well as for the differences?

8 Educational use of the competencies

Up until now, this article has only sporadically touched upon issues of mathematics education. Our focus so far has been to provide an up-to-date characterisation of what it means to be mathematically competent, from a cognitive point of view. However, our primary reason for addressing this question and for introducing a set of mathematical competencies into mathematics education was and is to provide a basis for the teaching and learning of mathematics—and for frameworks of both—that reflect the exercise and enactment of mathematics in more comprehensive and adequate ways than are typically encountered in research, development and practice of mathematics education. So, what do we, more specifically, gain from attributing key roles to mathematical competencies in educational contexts?

Firstly, the competencies can be used as a *normative* means for *designing curricula* in any context and level of mathematics education. By explicitly placing the emphasis on the enactment of mathematics at large, thus focusing not only on subject matter knowledge and procedural skills, competency oriented curriculum designs allow for a better balance between "doing" and "knowing" mathematics, as regards the aims, goals and the structure and organisation of the curriculum. The same is true of the design of modes and instruments of summative assessment, including tests and exams.

Secondly, the competencies can be used as an *analytic* means for *describing and characterising the state of affairs* concerning the competencies actually pursued (or not pursued) in a given segment of mathematics education, whether focusing on curricula, teaching, textbooks and other materials, modes and instruments of formative and summative assessment. It can further be used as a means for comparing and contrasting the state of affairs in different segments of the mathematics education system, including issues of transition from one segment to another.

Thirdly, the competencies can be used as a *diagnostic* means for designing ways to uncover and characterise the key elements of individual *students' learning of mathematics*, as manifested in their competency possession and development, and to detect mathematics specific learning difficulties, which is a crucial move in any reasonable attempt at remedying them.

Fourthly, the competencies can be used by *teachers* to *orchestrate, plan, carry out, monitor and evaluate their own teaching*, including student activities and assignments, both on a short and on a long term basis. In this context, designing instruments for assessing students' competencies is a significant undertaking.

Fifthly, the competencies can be used by *students* engaged in learning mathematics as a metacognitive support for *monitoring and controlling their own learning* activities and the outcomes of these by taking stock of the state and development of their competency possession.

It goes without saying that none of these forms of competency use is automatic. They all require thoughtfully and carefully designed, implemented and evaluated plans and activities, on the part of curriculum authorities, institutions, teachers and students.

9 Conclusion

In this article, we have given an updated account of the notions of mathematical competence and competencies, first introduced in the Danish KOM report (Niss & Højgaard, 2002; Niss & Højgaard, 2011), to pay due attention to a number of conceptual issues and developments that have arisen and taken place during the last couple of decades. The fundamentals of the initial ideas have, however, been preserved. Thus, we have maintained mathematical competence and competencies as basically cognitive constructs. In so doing, the significance of affective, dispositional and volitional factors of mathematical mastery and learning has in no way been disregarded, but these factors are of a different nature to the ones taken into account in this framework.

Of course, there is much more to say about mathematical competencies and their roles in mathematics education research, development and practice than has been said in this article. We are in the process of writing a follow up article about various research and development perspectives concerning the competency framework, and we are looking forward to offering a more comprehensive presentation and analysis of the framework in a forthcoming book (Niss & Højgaard, to appear).

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